JOURNAL OF DIFFERENTIAL EQUATIONS **134**, 134–147 (1997)
ARTICLE NO. DE963227

On Uniqueness in a Lateral Cauchy Problem with Multiple Characteristics*

Victor Isakov

Department of Mathematics and Statistics, Wichita State University, Wichita, Kansas 67260-0033

Received May 10, 1996

In this paper we consider the differential operators $\square_b$, Δ, $\partial_t^2 + \Delta^2$ and

order perturbations of related differential equations, e.g. for

$$\left(\partial_t^2 + \Delta^2 + \sum a_\alpha \partial^\alpha\right) u = 0 \quad \text{in} \quad Q$$
$$u = \cdots = \partial_\nu^3 u = 0 \quad \text{on} \quad \Gamma, \tag{0.1}$$

where Q is a cylindrical domain $\Omega \times (0, T)$ in $\mathbb{R}^{n+1}$ and Γ is a (cylindrical) part of its lateral boundary $\partial\Omega \times (0, T)$ such that *conv hull* $\Gamma = \Omega$. Here $|\alpha| \leqslant 2$, the coefficients a_α depend on the both time and space variables and are measurable and bounded. Actually, one of the goals of this study was to obtain similar uniqueness results for the (nonlinear) system describing a thin elastic plate

$$\partial_t^2 u - \operatorname{div}(\varepsilon(u) + f(\nabla w)) = 0 \quad \text{in} \quad Q$$
$$\partial_t^2 w - c\Delta\, \partial_t^2 w + \Delta^2 w - \operatorname{div}(\varepsilon(u) + f(\nabla w)) \cdot \nabla w = 0, \tag{0.2}$$

where $f(\nabla w) = 1/2 \nabla w \otimes \nabla w$. Indeed, we are able to derive from our results on scalar operators in this paper and in the paper [4] that a regular (C^5 is enough) solution to the system (0.2) is uniquely determined in the domain Q_o by the Cauchy data on Γ provided $f \in C^4$. The domain $Q_o \subset Q$ depends on Γ and is explicitly described by some quadratic function. We expect that other important systems of mathematical physics can be handled in a similar manner. As shown in [6, 7] uniqueness in the Cauchy

* This research was supported in part by NSF Grant DMS-9501510.

0022-0396/97 $25.00

Problem implies (for certain Γ) boundary controllability and also it plays an important role in inverse problems. We expect that the results of this paper about the Kirchhoff plate equations will lead to proofs of controllability and boundary stabilization of important applied systems considered in some cases in [2, 5].

We observe that both the equation (0.1) and the system (0.2) as well as the simpler scalar equation for the Kirchhoff plate considered in section 1 have multiple characteristics and can not be included in the existing theory of uniqueness in the Cauchy Problem [1, 4, 8]. Even the recent general and remarkable result of Tataru [9] can not be applied here.

Notation. $\partial_j = \partial/\partial x_j$, $x_o = t$, $x' = (x_1, ..., x_{n-1}, 0)$, α is the multiindex $(\alpha_o, ..., \alpha_n)$ with $|\alpha| = \alpha_o + \cdots + \alpha_n$, ∂^α is the differentiation $\partial_o^{\alpha_o} \cdots \partial_n^{\alpha_n}$, $\Box_c$ is $c\,\partial_o^2 - \Delta$ where Δ is the Laplace operator in the x-space $\mathbb{R}^n$. In this paper c is constant. Later on $\nabla^k u$ denotes $\{\partial^\alpha u \colon |\alpha| = k, \alpha_o = 0\}$. We will use the weighted norm

$$\|u\|_{\tau\phi}(Q) = \left(\int_Q |u|^2 e^{2\tau\phi}\right)^{1/2}.$$

In the proofs $\|\ \|_{\tau\phi}$ means $\|\ \|_{\tau\phi}(Q_\varepsilon)$ and $\|\ \|$ means $\|\ \|_o$. C denote (different) constants which might depend on Q and ε.

1. MAIN RESULTS

Introduce the weight function

$$\phi(x, t) = \exp(\lambda/2(x_1^2 + \cdots + x_{n-1}^2 + (x_n + a)^2 - \theta(t - T/2)^2 - s)) \tag{1.1}$$

and define the subdomain Q_ε of Q associated with ϕ as follows

$$Q_\varepsilon = Q \cap \{x_1^2 + \cdots + x_{n-1}^2 + (x_n + a)^2 - \theta(t - T/2)^2 - s > \varepsilon\}. \tag{1.2}$$

The parameters a, s will be chosen differently in the two following cases we are considering:

Case 1. the origin is contained in Ω, $\Gamma = (0, T) \times \partial\Omega$, then we let $a = 0$, $s = 0$

Case 2. Ω is a part of a domain Ω^* with the C^4-boundary and with the closure in the subspace $\{x_n < 0\}$, more precisely, $\Omega = \Omega^* \cap \{-H < x_n\}$, $\Gamma = (0, T) \times \gamma$, where $\gamma = \partial\Omega \cap \partial\Omega^*$, then we let $a > H$, $s = d^2 + (H - a)^2$ where d is sup $|x'|$ over $x \in \Omega$.

THEOREM 1.1. *Assume that*

$$0 < c, 0 < c\theta < 1. \tag{1.3}$$

Then for any large λ there is constant $C = C(\lambda)$ such that

$$\tau^{1/2}(\|\partial^\alpha w_o\|_{\tau\phi} + \|\partial_k \Delta w_o\|_{\tau\phi} + \|\partial_m \partial_o^2 w_o\|_{\tau\phi}) \leqslant C \|\Box_c \Delta w_o\|_{\tau\phi} \tag{1.4}$$

for all $w_o \in C_o^\infty(Q_\varepsilon)$ provided $|\alpha| \leqslant 2$, $k = 0, ..., n$, $m = 1, ..., n$ and $\tau > C$.

This so-called Carleman type estimate can be applied to different problems, in particular it implies the following uniqueness of the continuation result for the scalar equations of the Kirchhoff plate.

Consider the lateral Cauchy Problem

$$\partial_t^2 w - c\,\partial^2 \Delta w + \Delta^2 w + \sum b_\beta \partial^\beta \Delta w + \sum a_\alpha \partial^\alpha w = 0 \qquad \text{in} \quad Q \tag{1.5}$$

$$\partial_\nu^j w = g_j \quad \text{on } \Gamma,\ j = 0, ..., 3 \qquad \partial^\alpha w \in L^2(Q) \quad \text{when } |\alpha| \leqslant 4,\ \alpha_o \leqslant 2, \tag{1.6}$$

where the sums are over β, $|\beta| \leqslant 1$, α, $|\alpha| \leqslant 2$, and $a_\alpha, b_\beta \in L^\infty(Q)$.

THEOREM 1.2. *Assume that in Case 1 $\theta T^2 > 4r^2$ and in Case 2 $\theta T^2 > 4H(2a - H)$ and that the condition* (1.3) *is satisfied. Then a solution to the Cauchy Problem* (1.5), (1.6) *is unique in Q_o.*

We observe that in the Case 1 the uniqueness result is sharp because by choosing θ satisfying (1.3) and so that $c\theta$ is arbitrarily close to 1 we can derive from Theorem 1.2 uniqueness in the domain $Q \cap \{c\,|x|^2 > (t - T/2)^2\}$ which is the sharp result when T is close to $c^{1/2}2r$. In Case 2 the uniqueness domain is in more detail described in [3, section 1]. Although the result is not sharp anymore, the domain $(\Omega \times \{T/2\})_o$ is linearly approaching $\Omega \times \{T/2\}$ when $T \to \infty$.

Irena Lasiecka informed me that in some particular situations (special equations and domains) uniqueness results can be obtained by the multiplier method which also gives Lipschitz stability (and therefore exact controllability). More careful study of the Case 1 also will lead to Lipschitz stability results but we will not describe them in this paper.

A similar result is valid for the rather general system of partial differential equations describing an elastic shell

$$\begin{aligned} &\partial_t^2 u\text{- } \operatorname{div}(\varepsilon(u)) + L_1(x, t, w, \nabla w, \Delta w, \operatorname{div}(a \otimes \nabla w)) \\ &\qquad = 0 \qquad \text{in} \quad Q \subset \mathbb{R}^3 \end{aligned} \tag{1.7}$$

$$\partial_t^2 w - c\,\partial_t^2 \Delta w + \Delta^2 w + L_2(x, t, w, \nabla w, \Delta w, \operatorname{div}(a \otimes \nabla w), u, \nabla u, \nabla^2 u) = 0,$$

where L_1, L_2 are linear functions of $w, \nabla w, \operatorname{div}(\cdots), u, ..., \nabla^2 u$ with the coefficients in $C^3(\bar{Q})$, $a \in C^3(\bar{\Omega})$.

THEOREM 1.3. *A solution* $u \in H_{(4)}(Q)$, $\partial^\alpha w \in L^2(Q)$, $|\alpha| \leqslant 5$, $\alpha_o \leqslant 2$ *to the system* (1.7) *in* Q *with the given Cauchy Data* $\partial_\nu^j u = u_j$, $j = 0, 1, 2$, $\partial_\nu^m w = w_m$, $m = 0, ..., 3$, *on* Γ *is unique in* Q_o.

Uniqueness in the lateral Cauchy problem for the linear system (1.7) implies uniqueness of a solution $u \in C^4(\bar{Q})$, $w \in C^5(\bar{Q})$ of the Cauchy problem for the nonlinear system (0.3) because by subtracting the equations for two possible solutions, applying the (integral) mean value theorems to differences of nonlinear terms and denoting by u, w the differences of two solutions we obtain for u, w a system (1.7) and zero Cauchy data on Γ.

In the next results we let

$$\psi(x, t) = \exp(\lambda/2(x_1^2 + \cdots + x_{n-1}^2 + (x_n + a)^2 + (x_o - T/2)^2 - (a - H)^2), 0 < a \tag{1.8}$$

and consider the domain $Q = (0, T) \times \Omega$ in the case 2). Now we define

$$Q_\varepsilon = Q \cap \{(x_1^2 + \cdots + x_{n-1}^2 + (x_n + a)^2 + (x_o - T/2)^2 - (a - H)^2) > \varepsilon\}.$$

THEOREM 1.4. *There are large constants* λ, C *such that*

$$\tau^{1/2}(\|\partial^\alpha u_o\|_{\tau\psi} + \|\partial^\beta \Delta u_o\|_{\tau\psi}) \leqslant C\,\|(\partial_o^2 + \Delta^2)\, u_o\|_{\tau\psi} \tag{1.9}$$

for all $u_o \in C_o^\infty(Q_\varepsilon)$ *provided* $|\alpha| \leqslant 2$, $|\alpha_o| \leqslant 1$, $|\beta| \leqslant 1$, $\beta_o = 0$ *and* $\tau > C$.

Consider the Cauchy Problem

$$\partial_o^2 u + \Delta^2 u + \sum a_\beta\, \partial^\beta\, \Delta u + \sum a_\alpha\, \partial^\alpha u = 0 \qquad \text{in} \quad Q \tag{1.10}$$

$$\partial_\nu^j u = g_j \qquad \text{on} \quad \Gamma,\ j = 0, ..., 3, \tag{1.11}$$

where the sums are over $|\alpha| \leqslant 2$, $\alpha_o \leqslant 1$, $|\beta| \leqslant 1$, $\beta_o = 0$ and a_α are bounded measurable functions on Q and $\partial^\alpha u \in L^2(Q)$ when $2\alpha_o + \alpha_1 + \cdots + a_n \leqslant 4$.

THEOREM 1.5. *A solution* u *to the Cauchy Problem* (1.10), (1.11) *is unique on* Q.

2. PROOFS OF CARLEMAN TYPE ESTIMATES

We will derive Theorem 1.1 from the following results.

THEOREM 2.1. *For any large λ there is a constant $C(\lambda)$ such that*

$$\tau^{1/2}\|\partial^{\beta}v\|_{\tau\phi} \leqslant C\|\Delta v\|_{\tau\phi},\ v \in C_o^{\infty}(Q_{\varepsilon}) \tag{2.1}$$

provided $|\beta| \leqslant 1$, $\beta_o = 0$ and $\tau \geqslant C$.

Proof. First we will derive the Carleman estimates with the weight function ϕ on the n-dimensional domains $Q(s) = \Omega \times \{s\}$, $O < s < T$ by verifying pseudoconvexity of $\phi(, s)$ with respect to the x-Laplacian Δ. To do so we consider the Levi form

$$\mathscr{H}(\zeta) = \sum \partial_j \partial_k \phi P^{(j)}\,\overline{P^{(k)}} \text{ (the sum is over } j, k = 1, ..., n),$$

where $\zeta = \zeta + i\tau\nabla_x\phi$, $\xi \in \mathbb{R}^n$, $P^{(j)}(\zeta) = \partial P(\zeta)/\partial\zeta_j$ and $P(\zeta) = \zeta\cdot\zeta$. As in Lemma 3.1 of the paper [3] we have

$$\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2,$$

where

$$\mathscr{H}_1 = \lambda\phi(x, t)(|\zeta_1|^2 + \cdots + |\zeta_n|^2),\ \mathscr{H}_2 = \lambda^2\phi(x, t)|x_1\zeta_1 + \cdots + x_n\zeta_n|^2.$$

Hence $\mathscr{H}(\zeta) \geqslant \varepsilon_1 |\zeta|^2$ and we have the strict pseudoconvexity of ϕ and therefore the Carleman type estimate

$$\tau^{1/2}\int |\partial^{\beta}v(x, t)|^2\, e^{2\tau\phi(x,t)}\,dx \leqslant C\int |\Delta v|^2\,(x, t)\,e^{2\tau\phi(x,t)}\,dx,\ v \in C_o^{\infty}(Q_{\varepsilon}).$$

Integrating with respect to t over $(0, T)$ we obtain the estimate (2.1). We observe that a more careful analysis of the proofs in [1] and [4] shows that C depends only on ε_1 and on the continuity modulo of the function ϕ and its second derivatives, so it might be choosen t-independent.

The proof is complete.

THEOREM 2.2. *If a constant c satisfies the conditions* (1.3), *then for any large λ there is $C(\lambda)$ such that*

$$\lambda^{1/2}\tau^{1/2}\,\|\partial^{\beta}w^*\|_{\tau\phi} \leqslant C\,\|\Box_c w^*\|_{\tau\phi},\ w^* \in C_o^{\infty}(Q_{\varepsilon}) \tag{2.2}$$

provided $|\beta| \leqslant 1$ and $\tau > C$.

A proof of this result is given in [3], Theorem 1.1*, where there is a misprint: $\lambda^{3-2|\alpha|}$ must be replaced by $\lambda^{1/2}\tau^{3/2-|\alpha|}$.
Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will show that

$$\tau^{1/2}(\|\partial^\alpha w\|_{\tau\phi}+\|\partial_k \Delta w\|_{\tau\phi}+\|\Box_c w\|_{\tau\phi}+\|\partial_m \Box_c w\|_{\tau\phi})\leqslant C\,\|\Box_c \Delta w\|_{\tau\phi} \tag{2.3}$$

for all $w\in C_o^\infty(Q_\varepsilon)$ when $|\alpha|\leqslant 2$, $\alpha_o\neq 2$, $k=0, ..., n$, $m=1, ..., n$.
Since

$$\|\partial_o^2 w\|_{\tau\phi}=\|c^{-1}(\Box_c w-\Delta w)\|_{\tau\phi}\leqslant C(\|\Box_c w\|_{\tau\phi}+\|\Delta w\|_{\tau\phi})$$
$$\|\partial_m\,\partial_o^2 w\|_{\tau\phi}=\|\partial_m c^{-1}(\Box_c w-\Delta w)\|_{\tau\phi}\leqslant C(\|\partial_m \Box_c w\|_{\tau\phi}+\|\partial_m \Delta w\|_{\tau\phi})$$

the bound (1.4) follows from the bounds (2.3).
To prove (2.3) we first make use of the estimate (2.1) with $v=\Box_c w$ to obtain

$$\tau^{1/2}(\|\Box_c w\|_{\tau\phi}+\|\partial_m \Box_c w\|_{\tau\phi})\leqslant C\,\|\Box_c\,\Delta w\|_{\tau\phi}. \tag{2.4}$$

Similarly, from (2.2) with $w^*=\Delta w$ we get

$$\tau^{1/2}(\|\Delta w\|_{\tau\phi}+\|\partial_k\,\Delta w\|_{\tau\phi})\leqslant C\,\|\Box_c \Delta w\|_{\tau\phi}. \tag{2.5}$$

Applying again (2.2) with $w^*=w$, $w^*=\partial_m w$ we will have

$$\tau^{1/2}\,\|\partial^\alpha w\|_{\tau\phi}\leqslant C\left(\|\Box_c w\|_{\tau\phi}+\sum\|\partial_m \Box_c w\|_{\tau\phi}\right), \tag{2.6}$$

where the sum is over $m=1, ..., n$, and $|\alpha|\leqslant 2$, $\alpha_o\neq 2$.
Combining (2.4)–(2.6) gives (2.3).
The proof is complete.

A similar decomposition argument works for Theorem 1.4, but now instead of isotropic Carleman estimates in [1] we will use their anisotropic versions in [4].

Proof of Theorem 1.4. We will show that

$$\tau^{1/2}(\|\partial^\alpha u\|_{\tau\phi}+\|(i\partial_o-\Delta)\,u\|_{\tau\phi}+\|\partial^\beta(i\partial_o-\Delta)\,u\|_{\tau\phi})$$
$$\leqslant C\,\|(\partial_o^2+\Delta^2)\,u\|_{\tau\phi},\ u\in C_o^\infty(Q_\varepsilon) \tag{2.7}$$

when $|\alpha|\leqslant 2$, $\alpha_o=0$, $|\beta|\leqslant 1$, $\beta_o=0$. Since replacing t by $-t$ does not change properties of the Schrodinger operator the inequality (2.7) is valid also with $i\partial_o-\Delta$ replaced by $i\partial_o+\Delta$.

Since

$$\|\partial_o u\|_{\tau\phi} + \|\Delta u\|_{\tau\phi} \leqslant \|(i\partial_o + \Delta)\, u\|_{\tau\phi} + \|(i\partial_o - \Delta)\, u\|_{\tau\phi}$$

and we have this inequality with $\partial^\beta u$ instead of u, the estimate (2.7) and the estimate with $-i\partial_o$ instead of $i\partial_o$ imply (1.9).

To prove (2.7) we remind the Carleman-type estimate for the Schrodinger equation

$$\tau^{1/2}\, \|\partial^\beta v\|_{\tau\psi} \leqslant C\, \|(i\partial_o + \Delta)\, v\|_{\tau\psi} \tag{2.8}$$

provided $|\beta| \leqslant 1$, $\beta_o = 0$ and $\tau > C$. The estimate (2.8) follows from the results of the paper [4, Corollary 6.3 and its proof] and is valid when $+\Delta$ is replaced by $-\Delta$.

We make use of (2.8) with $v = (i\partial_o - \Delta)\, u$ to obtain

$$\tau^{1/2}(\|i\partial_o - \Delta)\, u\|_{\tau\psi} + \|\partial^\beta(i\partial_o - \Delta)\, u\|_{\tau\psi}) \leqslant C\, \|(\partial_o^2 + \Delta^2)\, u\|_{\tau\psi}. \tag{2.9}$$

Similarly we use (2.8) with $-\Delta$ to obtain this estimate with $+\Delta$ instead of $-\Delta$.

Applying again (2.8) with $v = \partial^\beta u$, $|\beta| \leqslant 1$, $\beta_o = 0$ (and with $-\Delta$ instead of Δ), we will have

$$\tau^{1/2}\, \|\partial^\alpha u\|_{\tau\psi} \leqslant C\Big(\, \|(i\partial_o - \Delta)\, u\|_{\tau\psi} + \sum \|\partial^\beta(i\partial_o - \Delta)\, u\|_{\tau\psi}\Big) \tag{2.10}$$

provided $|\alpha| \leqslant 2$, $\alpha_o = 0$.

Combining the both inequalities (2.9) (with $-\Delta$ and $+\Delta$) and (2.10) gives (2.7).

The proof is complete.

3. PROOFS OF UNIQUENESS IN THE CAUCHY PROBLEMS

Carleman estimates in some standard way can be applied to obtain uniqueness (and stability) in the lateral Cauchy problems. However we will repeat this argument in some detail because in our case it is not quite clear what "lower order terms" mean and if it is clear whether our Carleman estimates suffice to treat them.

In the proofs we will make use of the following well-known Leibniz' formula of the differentiation of product of functions

$$P(\chi w) = \sum \partial^\alpha \chi P^{(\alpha)} w/\alpha!, \text{ where } P^{(\alpha)}(\xi) = \partial_\xi^\alpha P(\xi),\ \alpha! = \alpha_o! \cdots \alpha_n! \tag{3.1}$$

Proof of Theorem 1.2. To prove uniqueness we assume that the Cauchy data for w are zero and will show that $w=0$ on $Q_{3\varepsilon}$ for any $\varepsilon>0$.

The Carleman estimates of Theorem 1.1 are valid on compactly supported functions. To use them we introduce a cut-off function $\chi\in C^\infty(\mathbb{R}^{n+1})$ which is 1 on $Q_{3\varepsilon}$ and zero on $Q_\varepsilon\backslash Q_{3\varepsilon}$. Since the Cauchy data of w are zero on Γ and the conditions on T guarantee that $\partial Q_\varepsilon\backslash Q\subset\Gamma$, the function $w_o=\chi w$ and its derivatives $\partial^\alpha w_o$, $|\alpha|\leqslant 4$, $\alpha_o\leqslant 2$, are the L^2-limits (as $\delta\to 0$) of $\partial^\alpha w_\delta$ where $w_\delta\in C^\infty(Q_\varepsilon)$. Hence the estimate (1.4) is valid for w_o.

From the Leibniz' formula (3.1) it follows that

$$\Box_c\,\Delta w_o=\chi\Box_c\,\Delta w+\sum b_k\,\partial_k\,\Delta w+\sum c_m\,\partial_m\,\partial_o^2 w+\sum a_\alpha\,\partial^\alpha w,$$

where the sums are over $k=0,\ldots,n$, $m=1,\ldots,n$, $|\alpha|\leqslant 2$, and b_k, c_m, $a_\alpha\in L^\infty(Q)$. Since w solves the equation (1.5) we conclude that

$$|\Box_c\,\Delta w_o|^2\leqslant C\left(\sum|\partial_k\,\Delta w|^2+\sum|\partial_m\,\partial_o^2 w|^2+\sum|\partial^\alpha w|^2\right)\quad\text{on}\quad Q_\varepsilon.$$

Multiplying the both parts by $e^{\tau\phi}$, integrating over Q_ε and using Theorem 1.1 (with some fixed λ) we get

$$\tau\left(\sum\|\partial^\alpha w_o\|^2_{\tau\phi}+\sum\|\partial_k\,\Delta w_o\|^2_{\tau\phi}+\sum\|\partial_m\,\partial_o^2 w_o\|^2_{\tau\phi}\right.$$
$$\left.\leqslant C\left(\sum\|\partial^\alpha w\|^2_{\tau\phi}+\sum\|\partial_k\,\Delta w\|^2_{\tau\phi}+\sum\|\partial_m\,\partial_o^2 w\|^2_{\tau\phi}\right)\right..$$

Shrinking the integration domain in the left side to $Q_{3\varepsilon}$, observing that $w_o=w$ there we replace in the inequality w_o by is and the $L^2(Q_\varepsilon)$-norms in the left side by the $L^2(Q_{3\varepsilon})$-norms. Splitting the integration domain in the right side into $Q_{3\varepsilon}$ and $Q_\varepsilon\backslash Q_{3\varepsilon}$ and choosing $\tau>2C$ we will cancel the integrals over $Q_{3\varepsilon}$ in the right side to obtain

$$\tau\left(\sum\|\partial^\alpha w\|^2_{\tau\phi}(Q_{3\varepsilon})+\sum\|\partial_k\,\Delta w\|^2_{\tau\phi}(Q_{3\varepsilon})+\sum\|\partial_m\partial_o^2 w\|^2_{\tau\phi}\,(Q_{3\varepsilon})\right)$$
$$\leqslant C\left(\sum\|\partial^\alpha w\|^2_{\tau\phi}(Q_\varepsilon\backslash Q_{3\varepsilon})+\sum\|\partial_k\,\Delta w\|^2_{\tau\phi}(Q_\varepsilon\backslash Q_{3\varepsilon})\right.$$
$$\left.+\sum\|\partial_m\,\partial_o^2 w\|^2_{\tau\phi}\,(Q_\varepsilon\backslash Q_{3\varepsilon})\right).$$

According to the definition of Q_ε we have $\phi \leqslant \Phi$ on $Q_\varepsilon \backslash Q_{3\varepsilon}$ and $\Phi < \psi$ on Q_ε where Φ is the value of ϕ at a joint point of closures of Q_ε and of $Q_{3\varepsilon} \backslash Q_\varepsilon$. Replacing ϕ in the integrals by Φ and diving the both parts of the last inequality by $e^{2\Phi}$ we finally arrive to the inequality

$$\tau\left(\sum \|\partial^\alpha w||^2(Q_{3\varepsilon}) + \sum \|\partial_k \Delta w\|^2 (Q_{3\varepsilon}) + \sum \|\partial_m \partial_o^2 w\|^2 (Q_{3\varepsilon})\right)$$

$$\leqslant C\left(\sum \|\partial^\alpha w\|^2 (Q_\varepsilon \backslash Q_{3\varepsilon}) + \sum \|\partial_k \Delta w\|^2 (Q_\varepsilon \backslash Q_{3\varepsilon})\right.$$

$$\left. + \sum \|\partial_m \partial_o^2 w\|^2 (Q_\varepsilon \backslash Q_{3\varepsilon})\right)$$

Letting $\tau \to \infty$ we conclude that $w = 0$ on $Q_{3\varepsilon}$.

The proof is complete.

A similar scheme will be used in the proof of Theorem 1.3 which is however much more complicated due to the structure of the (uncoupled) system of equations and to the use of quite strong estimates from [3] in addition to Theorem 1.1.

Proof of Theorem 1.3. To use the Carleman Estimates for scalar equations we will derive from our system (1.7) a new system whose "principal part" is scalar.

Let E be the linear elasticity matrix differential operator $\partial_t^2 I$-div(ε) where $\varepsilon_{jk}(u) = 1/2(\partial_k u_j + \partial_j u_k)$ is the stress tensor. Here $j, k = 1, 2$, and I is the identity matrix in $\mathbb{R}^2$. Observe that $E = 1/2(\Box_2 I - |\partial_j \partial_k|)$ and introduce the co-operator $E^{co} = (2\Box_1 I + |\partial_j \partial_k|)$. It is known (and easy to check) that $E^{co} E = \Box_2 \Box_1 I$.

Using these remarks and applying the operator E^{co} to the first two equations (1.7) and differentiating the last equation with respect to x_1, x_2 we arrive to the following system

$$\begin{aligned}
&\Box_2 \Box_1 u = L_3(x, t, w, \partial_o w, \nabla w, \partial_o^2 w, \partial_o \nabla w, \nabla^2 w, \\
&\qquad\qquad \nabla \partial_o^2 w, \nabla^2 \partial_o w, \nabla^3 w, \partial_o^2 \nabla^2 w, \nabla^2 \Delta w) \\
&\partial_t^2 w - c\, \partial_t^2 \Delta w + \Delta^2 w = L_4(x, t, u, \nabla u, \nabla^2 u, w, \nabla w, \nabla^2 w) \qquad \text{in} \quad Q \\
&\partial_t^2 \partial_m w - c\, \partial_t^2 \Delta \partial_m w + \Delta^2 \partial_m w = L_{4m}(x, t, u, ..., \nabla^3 u, w, ..., \nabla^3 w),
\end{aligned} \tag{3.2}$$

where $m = 1, 2$. In the first equations the dependence on $\nabla^4 w$ is specific due to the relations

$$|\partial_j \partial_k| \operatorname{div}(a \otimes \nabla w) = L(x, t, w, \nabla w, \nabla^2 w, \nabla^3 w, \nabla^2 \Delta w),$$

where L is a linear functions of $w, ..., \nabla^2 \Delta w$ with $L_\infty(Q)$-coefficients.

Bounding from (3.2) the left side of the first two equations by the sum of the terms in the right side we obtain

$$|\Box_2 \Box_1 u_j|^2 \leqslant C\left(\sum |\partial^\alpha w|^2 + \sum |\partial^\beta \partial_o^2 w|^2 + \sum |\partial^\beta \Delta w|^2\right), \tag{3.3}$$

where the sums are over $|\alpha| \leqslant 3$, $\alpha_o \leqslant 2$, $|\beta| \leqslant 2$, $\beta_o = 0$. Observe that $-c\,\partial_o^2 \Delta + \Delta^2 = -\Box_c \Delta$. Referring the terms $\partial_o^2 w$, $\partial_o^2 \partial_m w$ to the right side of the last three equations (3.2) we similarly obtain

$$|\Box_c \Delta w|^2 + |\Box_c \Delta \partial_m w|^2 \leqslant C\left(\sum |\partial^\alpha u_j|^2 + \sum |\partial^\beta w|^2\right) \quad \text{on} \quad Q, \tag{3.4}$$

where the sums are over α, β, such that $|\alpha| \leqslant 3$, $|\beta| \leqslant 3$, $\beta_o \leqslant 2$ and $j, m = 1, 2$.

Let χ be a $C^\infty(\mathbb{R}^3)$-function such that $\chi = 1$ on $Q_{3\varepsilon}$ and $\chi = 0$ on $Q_\varepsilon Q_{3\varepsilon}$. Since the Cauchy data of u, w are zero on Γ and due to the choice of T this surface is contained in ∂Q_o the functions $u_o = \chi u$, $w_o = \chi w$ and their derivatives $\partial^\alpha u_o$, $|\alpha| \leqslant 4$, and $\partial^\beta w_o$, $|\beta| \leqslant 5$, $\beta_o \leqslant 2$, are $L^2(Q_o)$-limits (as $\delta \to 0$) of $\partial^\alpha u_\delta$, $\partial^\beta w_\delta$ where u_δ, $w_\delta \in C_o^\infty(Q_\varepsilon)$-functions, so we can apply to them Carleman estimates of Theorem 1.1 and of the paper [3]. On other hand by using the Leibniz' formula (3.1) we can see that

$$\Box_2 \Box_1 u_{oj} = \chi \Box_2 \Box_1 u + \sum a_\alpha \partial^\alpha u, \ \Box_c \Delta w_o = \chi \Box_\gamma \Delta w + \sum b_\beta \partial^\beta w$$

$$\Box_c \Delta\, \partial_m w_o = \chi \Box_c \Delta \partial_m w + \sum c_\gamma \partial^\gamma \partial_o^2 w + \sum d_\delta \partial^\delta \Delta w + \sum b_\beta^* \partial^\beta w, \tag{3.5}$$

where the sums are over α, $|\alpha| \leqslant 3$, β, $|\beta| \leqslant 3$, $\beta_o \leqslant 2$, γ, $|\gamma| \leqslant 2$, $\gamma_o = 0$, δ, $|\delta| \leqslant 2$, $\delta_o \leqslant 1$, and a_α, b_β, c_γ and d_δ are some functions in $L^\infty(Q)$. The most delicate part is the formula for $\Box \Delta\, \partial_m w_o$ and we give some detail. From (3.1) with $P(\partial) = c\,\partial_o^2 \Delta\, \partial_1 - \partial_1 \Delta^2$ we have

$$P^{(1,1,1)} = 2\partial_o \partial_1 \Delta, \ P^{(0,1,0)} = c\,\partial_o^2 \Delta + 2c\,\partial_o^2 \partial_1^2 - \Delta^2 - 4\Delta\,\partial_1^2, \ P^{(0,0,1)}$$
$$= 4c\,\partial_o^2 \partial_1 \partial_2 - 4\Delta\,\partial_1 \partial_2$$

so the fourth order terms with respect to w are either $c_\gamma \partial^\gamma \partial_o^2 w$ or $d_\delta \partial^\delta \Delta w$. The third order terms do not involve t-differentiation of order higher than 2, so they have the form $b_\beta^* \partial^\beta w$.

From (3.3), (3.4) and (3.5) we have

$$|\Box_2 \Box_1 u_{oj}|^2 \leqslant CM, \tag{3.6}$$

where

$$M = \sum |\partial^\alpha u_j|^2 + \sum |\partial^\beta w|^2 + \sum |\partial^\delta \Delta w|^2 + \sum |\partial^\gamma \partial_o^2 w|^2$$

and the sums are over α, β, γ, δ, satisfying the conditions in (3.5), $j=1, 2$ and similarly

$$|\Box_c \Delta w_o|^2 + |\Box_c \Delta \partial_m w_o|^2 \leqslant CM. \tag{3.7}$$

Summing the inequalities (3.6) and (3.7), multiplying the result by $e^{2\tau\phi}$ and integrating over Q_ε we obtain

$$\sum \|\Box_2 \Box_1 u_{oj}\|^2_{\tau\phi} + \|\Box_c \Delta w_o\|^2_{\tau\phi} + \|\Box_c \Delta \nabla w_o\|^2_{\tau\phi}$$

$$\leqslant C\left(\sum \|\partial^\alpha u_j\|^2_{\tau\phi} + \sum \|\partial^\beta w\|^2_{\tau\phi} + \sum \|\partial^\delta \Delta w\|^2_{\tau\phi} + \sum \|\partial^\gamma \partial^2{}_o w\|^2_{\tau\phi}\right). \tag{3.8}$$

By Theorem 1.1 of [3] for any large λ there are C_o and $C_1(\lambda)$ such that

$$\lambda \sum \tau^{6-2|\alpha|} \|\partial^\alpha u_{oj}\|^2_{\tau\phi} + \tau \sum \|\partial_k \Box_2 u_{jo}\|^2_{\tau\phi} \leqslant C_o \|\Box_1 \Box_2 u_{jo}\|^2_{\tau\phi} \tag{3.9}$$

when $\tau \geqslant C_1(\lambda)$.

From (3.8), (3.9) and Theorem 1.1 (with large λ) we obtain

$$\sum (C_o^{-1} \lambda \tau^{6-2|\alpha|} \|\partial^\alpha u_{oj}\|^2_{\tau\phi} + C^{-1}(\lambda)\, \tau(\|\partial^\beta w_o\|^2_{\tau\phi}$$

$$+ \|\partial^\gamma \partial_o^2 w_o\|^2_{\tau\phi} + \|\partial^\delta \Delta w_o\|^2_{\tau\phi}))$$

$$\leqslant C \sum (\|\partial^\alpha u_j\|^2_{\tau\phi} + \|\partial^\beta w\|^2_{\tau\phi} + \|\partial^\gamma \partial_o^2 w\|^2_{\tau\phi} + \|\partial^\delta \Delta w\|^2_{\tau\phi}), \tag{3.10}$$

where the sums are over $j=1, 2$, $|\alpha| \leqslant 3$, $|\beta| \leqslant 3$, $\beta_o \leqslant 2$, $|\gamma| \leqslant 2$, $\gamma_o = 0$, $|\delta| \leqslant 2$, $\delta_o \leqslant 1$.

Choose (and fix)

$$\lambda > 2C_o C \tag{3.11}$$

and assume that $\tau > 1$.

As in the proof of Theorem 1.2 we shrink the integration domain in the norms of the left side of (3.10) to $Q_{3\varepsilon}$ where $u_o = u$ and $w_o = w$, split the integration domains in the integrals of the right side into $Q_{3\varepsilon}$ and $Q_\varepsilon \backslash Q_{3\varepsilon}$ and using the choice of λ in (3.11) and choosing $\tau > 2CC(\lambda)$ we absorb the integrals over $Q_{3\varepsilon}$ from the right side by half of the left side we obtain the inequality

$$\tau\left(\sum \|u_j\|^2_{\tau\phi}(Q_{3\varepsilon}) + \|w\|^2_{\tau\phi}(Q_{3\varepsilon})\right) \leqslant C \sum (\|\partial^\alpha u_j\|^2_{\tau\phi}(Q_\varepsilon \backslash Q_{3\varepsilon})$$

$$+ \|\partial^\beta w\|^2_{\tau\phi}(Q_\varepsilon \backslash Q_{3\varepsilon}) + \|\partial^\gamma \partial_o^2 w\|^2_{\tau\phi}(Q_\varepsilon \backslash Q_{3\varepsilon}) + \|\partial^\delta \Delta w\|^2_{\tau\phi}(Q_\varepsilon \backslash Q_{3\varepsilon})).$$

As in the proof of Theorem 1.2 replacing ϕ by its infimum Φ over $Q_{3\varepsilon}$ and supremum over $Q_\varepsilon \backslash Q_{3\varepsilon}$ which are equal and dividing the both parts by $e^{2\tau\Phi}$ we will have

$$\tau\left(\sum \|u_j\|^2(Q_{3\varepsilon}) + \|w\|^2(Q_{3\varepsilon})\right) \leqslant C \sum (\|\partial^\alpha u_j\|^2(Q_\varepsilon) + \|\partial^\beta w\|^2(Q_\varepsilon)$$

$$+ \|\partial^\gamma \partial_o^2 w\|^2(Q_\varepsilon) + \|\partial^\delta \Delta w\|^2(Q_\varepsilon)),$$

where $\|\,\|$ is the L^2-norm and we increased the integration domains in the right side. Letting $\tau \to \infty$ we conclude that $u_j = 0$, $w = 0$ on $Q_{3\varepsilon}$.

The proof is complete.

Proof of Theorem 1.5. To prove uniqueness we assume that the Cauchy Data are zero and will show that $u = 0$ on any $Q^\bullet$ with the closure in $Q \cup \Gamma$.

Any such $Q^\bullet$ is contained in the set $\{-H + \varepsilon^\bullet < x_n < 0, \varepsilon^\bullet < t < T - \varepsilon^\bullet\}$ for some positive $\varepsilon^\bullet$. We will make use of the substitution $t = t^*$, $x_j = x_j^*$, $1 \leqslant j \leqslant n-1$, $x_n = x_n^* + h(t)$ where h is a C^∞-smooth non-negative function such that $h = 0$ on $(\varepsilon^\bullet, T - \varepsilon^\bullet)$, $h(0) = h(T) = H$. Directly from the Chain Rule one can see that the form of the equation (1.10) in new $*$-variables is the same, while the surface Γ is "t-curved" to Γ^* for t near 0 and T so that $\Gamma \cap \{-H < x_n^*\} = \partial Q^* \cap \{-H < x_n^*\}$. Observe that $Q^\bullet$ remaines unchanged. From now on we will drop $*$.

Let $\varepsilon = 2/3(a - H)\,\varepsilon^\bullet + \varepsilon^\bullet 2/3$ then $Q^\bullet \subset Q_{3\varepsilon}$. Due to our definition of Q_ε we can find (large) a so that $\partial Q_\varepsilon \cap \partial Q \subset \Gamma$.

Let χ be a $C^\infty(\mathbb{R}^{n+1})$-function such that $\chi = 1$ on $Q_{3\varepsilon}$ and $\chi = 0$ on $Q_\varepsilon \backslash Q_{2\varepsilon}$. Since the Cauchy data are zero on $\partial Q_\varepsilon \cap \partial Q$ the function $u_o = \chi u$ and all its derivatives $\partial^\alpha u$, $\partial^\beta \Delta u$, $\Delta^2 u$, $\partial_t^2 u$ are the $L^2(Q_\varepsilon)$-limits of $\partial^\alpha u_\delta$, $\partial^\beta \Delta u_\delta$, $\Delta^2 u_\delta$, $\partial_o^2 u_\delta$ where u_δ are functions in $C_o^\infty(Q_\varepsilon)$. Therefore the estimate (1.9) is valid for our function u_o.

By using the Leibniz' formula (3.1) one can show that

$$(\partial_t^2 + \Delta^2)\, u_o = \chi(\partial_t^2 + \Delta^2)\, u + \sum a_\beta\, \partial^\beta \Delta u + \sum a_\alpha\, \partial^\alpha u,$$

where a_α are some bounded and measurable functions (depending on χ) and the sums are over the same α and β as in (1.10). Since u solves the equation (1.10) we have that

$$|(\partial_t^2+\Delta^2)\, u_o|^2 \leqslant C(\varepsilon)\left(\sum |\partial^\beta \Delta u|^2+\sum |\partial^\alpha u|^2\right) \qquad \text{on} \quad Q_\varepsilon.$$

By using Theorem 1.4 we conclude that

$$\tau\left(\sum \|\partial^\alpha u_o\|_{\tau\psi}^2+\sum \|\partial^\beta \Delta u_o\|_{\tau\psi}^2\right)\leqslant C\left(\sum \|\partial^\alpha u\|_{\tau\psi}^2+\sum \|\partial^\beta \Delta u\|_{\tau\psi}^2\right),$$

where the norms are over Q_ε. Shrinking the integration domains in the left side to $Q_{3\varepsilon}$, observing that $u_o=u$ there we replace in this inequality u_o by u and the $L^2(Q_\varepsilon)$-norms in the left side by the $L^2(Q_{3\varepsilon})$-norms. Splitting the integration domains in the right side into $Q_{3\varepsilon}$ and $Q_\varepsilon \backslash Q_{3\varepsilon}$ and choosing $\tau>2C$ we cancel the integrals over $Q_{3\varepsilon}$ in the right side to obtain

$$\tau\left(\sum \|\partial^\alpha u\|_{\tau\psi}^2(Q_{3\varepsilon})+\sum \|\partial^\beta \Delta u\|_{\tau\psi}^2(Q_{3\varepsilon})\right)$$

$$\leqslant 2C\left(\sum \|\partial^\alpha u\|_{\tau\psi}^2(Q_\varepsilon\backslash Q_{3\varepsilon})+\sum \|\partial^\beta \Delta u\|_{\tau\psi}^2(Q_\varepsilon\backslash Q_{3\varepsilon})\right).$$

By the definition $\psi<\Psi$ on $Q_\varepsilon\backslash Q_{3\varepsilon}$ and $\Psi<\psi$ on $Q_{3\varepsilon}$ where Ψ is the value of ψ at a joint point of closures of the both domains. Replacing ψ in the integrals by Ψ and dividing the both parts of the new inequality by $e^{2\tau\Psi}$ we finally arrive to the inequality

$$\tau\left(\sum \|\partial^\alpha u\|^2(Q_{3\varepsilon})+\sum \|\partial^\beta \Delta u\|^2(Q_{3\varepsilon})\right)$$

$$\leqslant C(\varepsilon)\left(\sum \|\partial^\alpha u\|^2(Q\backslash Q_{3\varepsilon})+\sum \|\partial^\beta \Delta u\|^2(Q\backslash Q_{3\varepsilon})\right).$$

Letting $\tau\to+\infty$ we obtain that $u=0$ on $Q_{3\varepsilon}$. Since $Q^\bullet\subset Q_{3\varepsilon}$ we have $u=0$ on $Q^\bullet$.

The proof is complete.

ACKNOWLEDGMENTS

The author thanks Irena Lasiecka and Walter Littman for their initiative to study the Cauchy Problem for equations of elastic shells and advices. This work was supported in part by the Institute of Mathematics and its Applications with funds provided by the NSF.

REFERENCES

1. L. Hormander, "Linear Partial Differential Operators," Springer, Berlin, 1976.
2. M. A. Horn and I. Lasiecka, Asymptotic behavior with respect to thickness of boundary stabilizing feedback for the Kirchhoff plate, *J. Differential Equations* **114** (1994), 396–433.
3. V. Isakov, A nonhyperbolic Cauchy problem for $\Box_b \Box_c$ and its applications to elasticity theory, *Communic. Pure Appl. Math.* **39** (1986), 747–767.
4. V. Isakov, Carleman type estimates in an anisotropic case and applications, *J. Differential Equations* **105** (1993), 217–238.
5. I. Lasiecka and R. Triggiani, Exact controllability of semilinear abstract systems with application to waves and plates boundary control, *Appl. Math. Optim.* **23** (1991), 109–154.
6. J. L. Lions, "Controllabilité exacte, perturbations et stabilization de systèmes distribués," Vol. 1, Masson, Paris, 1989.
7. W. Littman and S. Taylor, Smoothing evolution equations and boundary control theory, *J. Analyse Math.* **59** (1992), 117–132.
8. L. Nirenberg, Uniqueness in the Cauchy Problem for differential equations with constant leading coefficients, *Comm. Pure Appl. Math.* **10** (1957), 89–105.
9. D. Tataru, Unique continuation for solutions to PDE's: between Hormander's theorem and Holmgren's theorem, *Comm. Partial Differential Equations* **20** (1995), 855–884.